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STATIC ANALYSIS OF MODERATELY-THICK FINITE ANTISYMMETRIC ANGLE-PLY CYLINDRICAL PANELS AND SHELLS

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Abstract—The analytical (exact in the limit) or strong (or differential) form of solutions to the bench-mark problems of (i) axisymmetric angle-ply circular cylindrical panels of rectangular planform and (ii) circumferentially complete circular cylindrical shells, subjected to transverse load and with SS2-type simply-supported boundary conditions prescribed at the edges, are presented. The problems investigated, which were hitherto thought to be incapable of admitting analytical solutions, have been solved, utilizing a recently developed novel boundary-discontinuous double Fourier series approach, for three kinematic relations, which are extensions of those due to Sanders, Love and Donnell to the first-order shear deformation theory (FSDT). Numerical results presented for two-layer square antisymmetric angle-ply panels, which demonstrate good convergence, and show the effects of fiber orientation and thickness on the static response of these panels, should serve as baseline solutions (in the context of FSDT) for future comparison with various approximate weak forms of solutions with either local (e.g. finite element methods) or global supports (e.g. Raleigh–Ritz, Galerkin).

1. INTRODUCTION

Analysis of laminated circular (circumferentially) complete cylindrical shells (e.g. rocket motor cases, submersibles, nuclear reactors, pressure vessels, pipes, tubes, etc.) and circular cylindrical panels (open shallow shells used in aircraft fuselages, wings, ships, roofs, etc.) is of current interest because of their increasing use in aerospace, hydrospace, energy, chemical and other industrial applications. It is a common practice to analyze these laminated shell structures by using such popular numerical techniques as the finite element methods (FEM), because of their inherent complexities introduced by the bending–stretching and other coupling effects (Seide and Chaudhuri, 1987). Confidence in these approximate numerical techniques is dependent upon the closeness of agreement of the numerically predicted displacements and stresses of certain bench-mark problems with their analytical counterparts. Thin or moderately thick cross-ply and antisymmetric angle-ply circular cylindrical shells and panels, with certain types of simply-supported boundary conditions, are two excellent examples of such bench-mark problems, which have attracted considerable attention in recent years.

Closed-form solutions have been obtained for the case of circular cylindrical shells subjected to axisymmetric internal pressure with/without temperature changes by Zukas and Vinson (1971), Reuter (1972), Chaudhuri *et al.* (1986), and Abu-Arja and Chaudhuri (1989) and for axisymmetric buckling of such shells by Hirano (1979). Analytical (e.g. double Fourier series, which are either exact or exact in the limit) solutions have been presented by Stavsky and Lowey (1971), Dong and Tso (1972), Jones and Morgan (1975), Sinha and Rath (1976), Greenberg and Stavsky (1980), Hsu *et al.* (1981), Soldatos and Tzivanidis (1982), Bert and Kumar (1982) and Reddy (1984), for non-axisymmetric deformation, buckling and vibration of cross-ply circular (circumferentially) complete cylindrical shells and panels, with SS3-type [under the classification of Hoff and Rehfield (1965) and Chaudhuri *et al.* (1986)] simply-supported boundary conditions prescribed at the edges. Soldatos (1984) has used a Flugge-type theory and Galerkin's approach in solving the free vibration problem of non-circular cross-ply cylindrical shells.

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With regard to the problems of antisymmetric angle-ply cylindrical panels, Soldatos (1983a,b) and Whitney (1984) have resorted to Galerkin's method. Soldatos (1983b) attributes the non-existence of an exact solution for an angle-ply cylindrical panel to the complexity of the geometry of a shell (as compared to a flat plate) besides the material properties, by concluding, "thus, for equations of motion of antisymmetric angle-ply laminated circular cylindrical panels and shells whose geometry is more complicated an exact closed-form solution seems to be impossible". While this statement does not attribute impossibility of obtaining other types of exact (i.e. in the limit) or analytical solutions, no such solutions have been attempted by either Soldatos or any other researcher, thus leaving a critical analytical vacuum. Reddy (1984) has succeeded in obtaining Navier's solution in the form of double Fourier series to the problems of cross-ply laminated shells, and has concluded that "Closed-form solutions for deflections and natural frequencies of simply supported, cross-ply laminated ... shells are derived .... Unlike plates, antisymmetric angle-ply laminated shells with simply-supported boundary conditions do not admit exact solutions. The exact solutions presented herein for cylindrical and spherical cross-ply shells under sinusoidal, uniformly distributed, and point loads should serve as benchmark results for approximate methods, such as finite element and finite difference methods". It may be pointed out here that there appears to be some confusion or controversy with regard to the definition of "exact solution" in the literature. While expressions for eigenvalues, e.g. natural frequencies, obtained by both Reddy (1984) and Soldatos and Tzivanidis (1982) for cross-ply shells are in closed form, the solutions for deflection presented by the former are definitely not in closed form. Reddy's (1984) solutions for deflection of cross-ply shells can only be regarded as solutions of strong (or differential) form, or solutions exact in the limit or exact in the sense of Chia (1980, p. 38) and Szilard (1974, p. 43). Chia (1980) has stated, "The solution is said to be exact in the sense that an infinite set of ... algebraic equations can be truncated to obtain any desired degree of accuracy. Exact solutions will be obtained by double Fourier series, generalized double Fourier series and a combination of these series ...". According to Szilard (1974), "In general, there are four types of mathematically 'exact' solutions ...: 1. Closed-form solution. 2 ... 3. Double trigonometric series solution. 4. Single series solution".

Recently, Chaudhuri (1989) has presented a novel method for obtaining an analytical [exact in the limit or exact in the sense of Chia (1980) and Szilard (1974)] or strong form of solution, in the form of boundary-discontinuous double Fourier series, to the problem of a system of completely coupled linear partial differential equations with constant coefficients, subjected to completely coupled boundary conditions. This method (i) provides guidance with regard to the selection of appropriate assumed double Fourier series solution functions, depending on the coefficients of the system of governing partial differential equations, (ii) guides in making decisions with regard to the discontinuities either in the assumed solution functions or their first derivatives, depending on the coefficients of the boundary condition equations, (iii) ensures uniqueness of the solution, and finally, (iv) leads to a highly efficient computational scheme in spite of the complexity of the completely (or highly) coupled PDEs. Chaudhuri (1987, 1989) has also applied this method to investigate the general nature of these exact double Fourier series solutions in the case of moderately-thick doubly-curved laminated anisotropic shells of rectangular planform. Analytical solutions to the problems of antisymmetric angle-ply cylindrical panels and circumferentially complete cylindrical shells have, however, not been investigated in detail and their numerical results are still non-existent.

The primary objective of the present study is to apply the aforementioned technique in obtaining a unique solution to the five highly coupled second-order PDEs subjected to highly coupled boundary conditions. This study will present (i) analytical (double Fourier series, which are exact in the limit) or strong (or differential) forms of solutions to the aforementioned bench-mark problems for three kinematic relations (shell theories), which are extensions, by Bert and Kumar (1982), of those due to Donnell, Love and Sanders to the case of first-order shear deformation theory (FSDT) and (ii) some useful numerical results for antisymmetric angle-ply cylindrical panels limited to Sanders' kinematic relations alone, as a first step. Although the problem of doubly-curved angle-ply panels was solved by Chaudhuri and Abu-Arja (1988) using a preliminary version (Chaudhuri, 1987) of the boundary-discontinuous double Fourier approach due to Chaudhuri (1989), the criteria determining when the boundary Fourier series are needed or not needed were neither fully understood, in the mathematical sense, by the authors at that time, nor were ever fully investigated by such previous investigators as Goldstein (1936, 1937), Green (1944), Green and Hearmon (1945), Whitney (1970, 1971) and Whitney and Leissa (1970). This issue has now been completely investigated, in the mathematical sense, by Chaudhuri (1989). The present analysis on antisymmetric angle-ply cylindrical panels and shells is, therefore, firmly rooted to the mathematical analysis, while keeping the door open to physical interpretation by the intuitive approach of the aforementioned earlier investigations. The numerical results presented here are expected to serve as bench-mark solutions (in the context of FSDT) for future comparison with various approximate weak (or integral) forms of solutions with either local (e.g. finite element methods) or global supports (Raleigh–Ritz, Galerkin). Definitions of strong and weak forms are available in Hughes (1987). The scope of the present study will be limited to the type of prescribed boundary conditions (i.e. SS2) considered by Soldatos (1983b) and Reddy (1984) for antisymmetric angle-ply panels, although the remaining boundary conditions can be handled with almost equal ease.

2. STATEMENT OF THE PROBLEM

For a cylindrical panel shown in Fig. 1, with axial and circumferential lengths, a and b, respectively, and radius, R, the strain–displacement relations, under FSDT (Reissner-Mindlin hypothesis) are

$$\varepsilon_1 = \varepsilon_1^0 + x_3\kappa_1;\quad \varepsilon_2 = \varepsilon_2^0 + x_3\kappa_2;\quad \varepsilon_4 = \varepsilon_4^0;\quad \varepsilon_5 = \varepsilon_5^0;\quad \varepsilon_6 = \varepsilon_6^0 + x_3\kappa_6 \tag{1}$$

where

$$\varepsilon_1^0 = u_{1,1};\quad \varepsilon_2^0 = u_{2,2} + u_3/R;\quad \varepsilon_4^0 = u_{3,2} + \phi_2 - \bar{c}_1 u_2/R;\quad \varepsilon_5^0 = u_{3,1} + \phi_1;$$
$$\varepsilon_6^0 = u_{1,2} + u_{2,1};\quad \kappa_1 = \phi_{1,1};\quad \kappa_2 = \phi_{2,2};\quad \kappa_6 = \phi_{2,1} + \bar{c}_2(u_{2,1} - u_{1,2})/R \tag{2}$$

in which a comma denotes partial differentiation. ε_i and ε_i^0 ($i = 1, 2, 6$) represent the surface-parallel normal and shearing strain components at a parallel surface and mid-surface respectively, while ε_i, ε_i^0 ($i = 4, 5$) represent the corresponding transverse shearing strain components. κ_i ($i = 1, 2, 6$) denotes the changes of curvature and twist, while u_i ($i = 1, 2, 3$) and ϕ_i ($i = 1, 2$) denote the displacement and rotation respectively in the ith direction. The coefficients ($\bar{c}_1, \bar{c}_2$) are shell theory tracers, for the extensions of Sanders', Love's and Donnell's theories respectively to their FSDT counterparts (Bert and Kumar, 1982). The equations of equilibrium are:

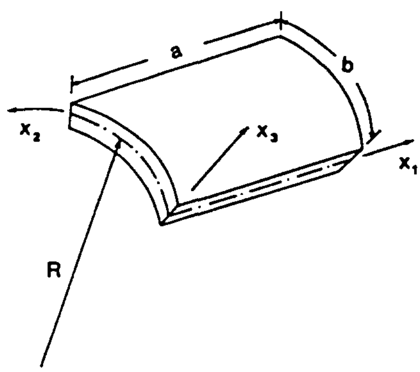

Fig. 1. A circular cylindrical panel.

$$N_{1,1}+N_{6,2}-\bar{c}_2 M_{6,2}/R = 0; \quad N_{6,1}+\bar{c}_2 M_{6,1}/R+N_{2,2}+\bar{c}_1 Q_2/R = 0;$$
$$N_2/R-Q_{1,1}-Q_{2,2}-q = 0; \quad M_{1,1}+M_{6,2}-Q_1 = 0; \quad M_{6,1}+M_{2,2}-Q_2 = 0 \tag{3}$$

in which q is the transverse or radial distributed load. Surface-parallel stress resultants, N_i, stress couples (moment resultants), M_i, and transverse shear stress resultants, Q_i, are related to the mid-surface strains, ε_i^0, and changes of curvature and twist, κ_i, by

$$N_i = A_{ij}\varepsilon_j^0+B_{ij}\kappa_j \quad (i,j = 1,2,6); \quad M_i = B_{ij}\varepsilon_j^0+D_{ij}\kappa_j \quad (i,j = 1,2,6);$$
$$Q_1 = A_{45}\varepsilon_4^0+A_{55}\varepsilon_5^0; \quad Q_2 = A_{44}\varepsilon_4^0+A_{45}\varepsilon_5^0. \tag{4}$$

Here A_{ij}, B_{ij}, D_{ij} are extensional, coupling, and bending rigidities, respectively, and A_{ij} $(i,j = 4,5)$ represents transverse shear rigidities. For an antisymmetric angle-ply laminate,

$$A_{16} = A_{26} = A_{45} = B_{11} = B_{12} = B_{22} = B_{66} = D_{16} = D_{26} = 0. \tag{5}$$

Substitution of eqns (2), (4), (5) into eqns (3) will yield five coupled partial differential equations with constant coefficients in the following form:

$$a_1^{(i)}u_1+a_2^{(i)}u_{1,11}+a_3^{(i)}u_{1,12}+a_4^{(i)}u_{1,22}+a_5^{(i)}u_2+a_6^{(i)}u_{2,11}+a_7^{(i)}u_{2,12}+a_8^{(i)}u_{2,22}+a_9^{(i)}u_{3,1}$$
$$+a_{10}^{(i)}u_{3,2}+a_{11}^{(i)}\phi_1+a_{12}^{(i)}\phi_{1,11}+a_{13}^{(i)}\phi_{1,12}+a_{14}^{(i)}\phi_{1,22}+a_{15}^{(i)}\phi_2+a_{16}^{(i)}\phi_{2,11}$$
$$+a_{17}^{(i)}\phi_{2,12}+a_{18}^{(i)}\phi_{2,22} = 0; \quad i = 1,2,4,5 \tag{6a}$$

$$a_1^{(3)}u_{1,1}+a_2^{(3)}u_{1,2}+a_3^{(3)}u_{2,1}+a_4^{(3)}u_{2,2}+a_5^{(3)}u_3+a_6^{(3)}u_{3,11}+a_7^{(3)}u_{3,12}+a_8^{(3)}u_{3,22}$$
$$+a_9^{(3)}\phi_{1,1}+a_{10}^{(3)}\phi_{1,2}+a_{11}^{(3)}\phi_{2,1}+a_{12}^{(3)}\phi_{2,2} = q \tag{6b}$$

where the superscripts of the coefficients, i, denote the equation number. $a_j^{(i)}$ $(i = 1,\ldots,5;$ $j = 1,\ldots,18)$ are as defined in Appendix B. The five boundary conditions at an edge are selected to be one member from each pair of the following:

$$(u_n, N_n) = (u_t, N_t) = (u_3, Q_n) = (\phi_n, M_n) = (\phi_t, M_t) = 0 \text{ at an edge } x_n = 0. \tag{7}$$

The boundary conditions considered here are the same as those considered by Soldatos (1983b) and Reddy (1984), which are termed SS2-type under the classification of Hoff and Rehfield (1965) and Chaudhuri *et al.* (1986). They are prescribed as follows:

$$u_1(0,x_2) = N_6(0,x_2) = u_3(0,x_2) = M_1(0,x_2) = \phi_2(0,x_2) = 0 \text{ at the edge } x_1 = 0. \tag{8}$$

3. SOLUTION FOR CYLINDRICAL PANELS

It has been shown by Chaudhuri (1989) that selection of the assumed boundary-discontinuous double Fourier series solution functions will depend on the governing partial differential equations and not the boundary conditions. At stake here is the well-posedness of the Fourier analysis, to be achieved through selection of the unknown coefficients of the assumed double Fourier series solution functions and introduction of certain boundary-discontinuous coefficients, so that the number of final algebraic equations become equal to the number of unknowns to furnish a unique complete solution. The details are available in Chaudhuri (1989) and will, in the interest of brevity, be excluded here. The solution to the system of five coupled partial differential equations, given by eqns (6) in conjunction with the SS2-type simply-supported boundary conditions, represented by eqns (8), will then be assumed in the form:

$$u_1 = u_1^{\mathrm{I}}+u_1^{\mathrm{II}}; \quad u_2 = u_2^{\mathrm{I}}+u_2^{\mathrm{II}}; \quad u_3 = u_3^{\mathrm{I}}+u_3^{\mathrm{II}}; \quad \phi_1 = \phi_1^{\mathrm{I}}+\phi_1^{\mathrm{II}}; \quad \phi_2 = \phi_2^{\mathrm{I}}+\phi_2^{\mathrm{II}} \tag{9}$$

where

$$(u_1^{\mathrm{I}}, u_2^{\mathrm{II}}, \phi_1^{\mathrm{II}}, \phi_2^{\mathrm{I}}) = \sum_{m=1}^{\infty} \sum_{n=0}^{\infty} (U_{mn}^{\mathrm{I}}, V_{mn}^{\mathrm{II}}, X_{mn}^{\mathrm{II}}, Y_{mn}^{\mathrm{I}}) \sin(\alpha_m x_1) \cos(\beta_n x_2);$$

$$(u_1^{\mathrm{II}}, u_2^{\mathrm{I}}, \phi_1^{\mathrm{I}}, \phi_2^{\mathrm{II}}) = \sum_{m=0}^{\infty} \sum_{n=1}^{\infty} (U_{mn}^{\mathrm{II}}, V_{mn}^{\mathrm{I}}, X_{mn}^{\mathrm{I}}, Y_{mn}^{\mathrm{II}}) \cos(\alpha_m x_1) \sin(\beta_n x_2);$$

$$u_3^{\mathrm{I}} = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} W_{mn}^{\mathrm{I}} \sin(\alpha_m x_1) \sin(\beta_n x_2);$$

$$u_3^{\mathrm{II}} = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} W_{mn}^{\mathrm{II}} \cos(\alpha_m x_1) \cos(\beta_n x_2); \tag{10}$$

in which

$$\alpha_m = m\pi/a; \quad \beta_n = n\pi/b. \tag{11}$$

The above equations introduce $(10mn+5m+5n+1)$ unknown Fourier coefficients. The next operation will comprise differentiation of the assumed solution functions, which is a necessary step before substitution into the equilibrium eqns (6). The procedure for differentiation of the assumed double Fourier series solution functions for the most general type of boundary condition has been described in detail by Chaudhuri (1989), the application of which to the present problem is relatively straightforward. The illustration of the procedure, described in Appendix A, will, in the interest of brevity of presentation, be limited to obtaining the first partial derivatives of the assumed solution function, u_1^{II}. These two derivatives directly follow from eqns (A2b), (A2c), (A8) and (A11) and can be rewritten, in the final form:

$$u_{1,1}^{\mathrm{II}}(x_1, x_2) = -\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} U_{mn}^{\mathrm{II}} \alpha_m \sin(\alpha_m x_1) \sin(\beta_n x_2) \quad (0 < x_1 < a, 0 < x_2 < b) \tag{12a}$$

$$u_{1,2}^{\mathrm{II}}(x_1, x_2) = \tfrac{1}{4}c_0 + \sum_{m=1}^{\infty} \tfrac{1}{2}c_m \cos(\alpha_m x_1) + \sum_{n=1}^{\infty} \{\beta_n U_{0n}^{\mathrm{II}} + \tfrac{1}{2}(c_0\gamma_n + d_0\delta_n)\} \cos(\beta_n x_2)$$
$$+ \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} (\beta_n U_{mn}^{\mathrm{II}} + c_m\gamma_n + d_m\delta_n) \cos(\alpha_m x_1) \cos(\beta_n x_2); \quad 0 \leqslant x_1 \leqslant a, 0 \leqslant x_2 \leqslant b \tag{12b}$$

where

$$(\gamma_i, \delta_i) = \begin{cases} (0, 1) \text{ for } i \text{ odd} \\ (1, 0) \text{ for } i \text{ even} \end{cases} \tag{13}$$

and

$$c_m = \frac{4}{ab} \int_0^a \{u_{1,1}^{\mathrm{II}}(x_1, b) - u_{1,1}^{\mathrm{II}}(x_1, 0)\} \cos(\alpha_m x_1)\, \mathrm{d}x_1 \tag{14a}$$

$$d_m = -\frac{4}{ab} \int_0^a \{u_{1,1}^{\mathrm{II}}(x_1, b) + u_{1,1}^{\mathrm{II}}(x_1, 0)\} \cos(\alpha_m x_1)\, \mathrm{d}x_1. \tag{14b}$$

Extension of the above procedure to the second derivatives is straightforward (Chaudhuri, 1989) and will yield:

$$u^{\text{II}}_{1,11}(x_1,x_2) = \frac{1}{2}\sum_{m=1}^{\infty} a_n \sin(\beta_n x_2) + \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}(-\alpha_m^2 U^{\text{II}}_{mn} + a_n\gamma_m + b_n\delta_m) \times \cos(\alpha_m x_1)\sin(\beta_n x_2) \quad (15a)$$

$$u^{\text{II}}_{1,12}(x_1,x_2) = -\frac{1}{2}\sum_{m=1}^{\infty} \alpha_m c_m \sin(\alpha_m x_1) - \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\alpha_m(\beta_n U^{\text{II}}_{mn} + c_m\gamma_n + d_m\delta_n) \times \sin(\alpha_m x_1)\cos(\beta_n x_2) \quad (15b)$$

$$u^{\text{II}}_{1,22}(x_1,x_2) = -\sum_{n=1}^{\infty}\{\beta_n^2 U^{\text{II}}_{0n} + \tfrac{1}{2}\beta_n(c_0\gamma_n + d_0\beta_n)\}\sin(\beta_n x_2) - \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \times \{\beta_n^2 U^{\text{II}}_{mn} + \beta_n(c_m\gamma_n + d_m\delta_n)\}\cos(\alpha_m x_1)\sin(\beta_n x_2) \quad (15c)$$

where

$$a_n = \frac{4}{ab}\int_0^b \{u^{\text{II}}_{1,1}(a,x_2) - u^{\text{II}}_{1,1}(0,x_2)\}\sin(\beta_n x_2)\,\mathrm{d}x_2 \quad (16a)$$

$$b_n = -\frac{4}{ab}\int_0^b \{u^{\text{II}}_{1,1}(a,x_2) + u^{\text{II}}_{1,1}(0,x_2)\}\sin(\beta_n x_2)\,\mathrm{d}x_2. \quad (16b)$$

A similar procedure applied to the other assumed functions leads to 16 more constants: e_n, f_n, i_n, j_n, m_n, n_n, q_n, r_n, for each n, the four pairs being associated with u^{II}_2, $u^{\text{II}}_{3,1}$, ϕ^{II}_1, $\phi^{\text{II}}_{2,1}$, respectively along the boundaries $x_1 = 0, a$; and g_m, h_m, k_m, l_m, o_m, p_m, s_m, t_m, for each m, the four pairs being associated with $u^{\text{II}}_{2,2}$, $u^{\text{II}}_{3,2}$, $\phi^{\text{II}}_{1,2}$, ϕ^{II}_2, respectively along the boundaries $x_2 = 0, b$. It may be noted that the first and second derivatives of the first parts of the assumed solution functions—u^{I}_i ($i = 1,2,3$) and ϕ^{I}_i ($i = 1,2$)—can be obtained by termwise differentiation (Chaudhuri, 1989). The details will be omitted here in the interest of brevity of presentation. The above step introduces ($10m + 10n + 12$) additional unknown coefficients, which ask for as many equations, to be supplied by the boundary conditions to be presented later. It will be interesting to offer the following physical explanation of the above mathematical operations, as applied to the present problem:

The first parts of the assumed solution functions u^{I}_i ($i = 1,2,3$) and ϕ^{I}_i ($i = 1,2$) which are the same as those assumed by Reddy (1984) as complete solutions, satisfy the prescribed boundary conditions, given by eqns (8), *a priori*. Since they do not violate any physical conditions at the boundary, their partial derivatives can be obtained by termwise differentiation. The second parts by themselves do not satisfy the prescribed boundary conditions. Hence, according to Chaudhuri's (1989) method, they are forced to satisfy the prescribed boundary conditions given by eqns (8). Still, discontinuities may arise out of violation of other physical conditions at certain boundaries by these functions and/or their first derivative(s). In such events, termwise differentiation of the assumed functions and/or their first partial derivative(s) concerned may not be valid, as stipulated by Green (1944), Green and Hearmon (1945), Winslow (1951), Whitney and Leissa (1969), Whitney (1971) and Chaudhuri (1989). The corresponding first or second partial derivative(s) must be obtained by expanding them in the form of double Fourier series, in a manner suggested by the above investigators. This will be illustrated for the case of the two first partial derivatives of the assumed solution function, u^{II}_1, which does not automatically satisfy the prescribed geometric boundary conditions at the edges $x_1 = 0, a$; these are then satisfied by force, which will yield additional equations arising out of satisfying boundary conditions to be described later. u^{II}_1 however, vanishes at the edges $x_2 = 0, b$, which is a violation, because at these edges $u_t = u_1$ and $N_t = N_6$ cannot, according to the variational principle, be simultaneously prescribed. Therefore, $u^{\text{II}}_{1,2}$ cannot be obtained by termwise differentiation, although $u^{\text{II}}_{1,1}$ can be obtained that way.

Expansion of the uniformly distributed transverse load into a double Fourier series

$$q = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} q_{mn}\sin(\alpha_m x_1)\sin(\beta_n x_2) \quad (17)$$

and substitution of the assumed solution functions and their appropriate partial derivatives into eqns (6a) will yield:

$$\begin{aligned}
&\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \sin(\alpha_m x_1)\cos(\beta_n x_2)\{(a_1^{(i)}-\alpha_m^2 a_2^{(i)}-\beta_n^2 a_4^{(i)})U^{I}_{mn} - \alpha_m\beta_n a_7^{(i)} V^{I}_{mn} + \beta_n a_{10}^{(i)} W^{I}_{mn}\\
&\quad - \alpha_m\beta_n a_{13}^{(i)} X^{I}_{mn} + Y^{I}_{mn}(a_{15}^{(i)} - \alpha_m^2 a_{16}^{(i)} - \beta_n^2 a_{18}^{(i)}) - \alpha_m\beta_n U^{II}_{mn} a_3^{(i)} - \alpha_m c_m\gamma_n a_3^{(i)}\\
&\quad - \alpha_m d_m\delta_n a_3^{(i)} + V^{II}_{mn}(a_5^{(i)} - \alpha_m^2 a_6^{(i)} - \beta_n^2 a_8^{(i)}) - \alpha_m e_n\gamma_m a_6^{(i)} - \alpha_m f_n\delta_m a_6^{(i)} + g_m\gamma_n a_8^{(i)}\\
&\quad + h_m\delta_n a_8^{(i)} - \alpha_m W^{II}_{mn} a_9^{(i)} + X^{II}_{mn}(a_{11}^{(i)} - \alpha_m^2 a_{12}^{(i)} - \beta_n^2 a_{14}^{(i)}) - \alpha_m m_n\gamma_m a_{12}^{(i)} - \alpha_m n_n\delta_m a_{12}^{(i)}\\
&\quad + o_m\gamma_n a_{14}^{(i)} + p_m\delta_n a_{14}^{(i)} - \alpha_m\beta_n Y^{II}_{mn} a_{17}^{(i)} - \alpha_m a_{17}^{(i)} s_m\gamma_n - \alpha_m a_{17}^{(i)} t_m\delta_n\} = 0
\end{aligned} \tag{18a}$$

$$\begin{aligned}
&\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \cos(\alpha_m x_1)\sin(\beta_n x_2)\{-\alpha_m\beta_n a_7^{(i)} U^{I}_{mn} + V^{I}_{mn}(a_5^{(i)} - \alpha_m^2 a_6^{(i)} - \beta_n^2 a_8^{(i)}) + \alpha_m a_9^{(i)} W^{I}_{mn}\\
&\quad + X^{I}_{mn}(a_{11}^{(i)} - \alpha_m^2 a_{12}^{(i)} - \beta_n^2 a_{14}^{(i)}) - \alpha_m\beta_n a_{17}^{(i)} Y^{I}_{mn} + U^{II}_{mn}(a_1^{(i)} - \alpha_m^2 a_2^{(i)} - \beta_n^2 a_4^{(i)})\\
&\quad + a_n\gamma_m a_2^{(i)} + b_n\delta_m a_2^{(i)} - \beta_n c_m\gamma_n a_4^{(i)} - \beta_n d_m\delta_n a_4^{(i)} - a_7^{(i)}\alpha_m\beta_n V^{II}_{mn} - a_7^{(i)}\beta_n e_n\gamma_m - a_7^{(i)}\beta_n f_n\delta_m\\
&\quad - \beta_n a_{10}^{(i)} W^{II}_{mn} - \alpha_m\beta_n a_{13}^{(i)} X^{II}_{mn} - a_{13}^{(i)}\beta_n m_n\gamma_m - a_{13}^{(i)}\beta_n n_n\delta_m + Y^{II}_{mn}\\
&\quad \times(a_{15}^{(i)} - \alpha_m^2 a_{16}^{(i)} - \beta_n^2 a_{18}^{(i)}) + q_n\gamma_m a_{16}^{(i)} + r_n\delta_m a_{16}^{(i)} - \beta_n s_m\gamma_n a_{18}^{(i)} - \beta_n t_m\delta_n a_{18}^{(i)}\} = 0
\end{aligned} \tag{18b}$$

$$\begin{aligned}
&\sum_{m=1}^{\infty} \sin(\alpha_m x_1)\{(a_1^{(i)} - \alpha_m^2 a_2^{(i)})U^{I}_{m0} + Y^{I}_{m0}(a_{15}^{(i)} - \alpha_m^2 a_{16}^{(i)}) - \tfrac{1}{2}\alpha_m c_m a_3^{(i)} + V^{II}_{m0}(a_5^{(i)} - \alpha_m^2 a_6^{(i)})\\
&\quad - \tfrac{1}{2}\alpha_m e_0\gamma_m a_6^{(i)} - \tfrac{1}{2}\alpha_m f_0\delta_m a_6^{(i)} + \tfrac{1}{2}g_m a_8^{(i)} - \alpha_m W^{II}_{m0} a_9^{(i)} + X^{II}_{m0}(a_{11}^{(i)} - \alpha_m^2 a_{12}^{(i)})\\
&\quad - \tfrac{1}{2}\alpha_m m_0\gamma_m a_{12}^{(i)} - \tfrac{1}{2}\alpha_m n_0\delta_m a_{12}^{(i)} + \tfrac{1}{2}o_m a_{14}^{(i)} - \tfrac{1}{2}a_{17}^{(i)}\alpha_m s_m\} = 0
\end{aligned} \tag{18c}$$

$$\begin{aligned}
&\sum_{n=1}^{\infty} \sin(\beta_n x_2)\{V^{I}_{0n}(a_5^{(i)} - \beta_n^2 a_8^{(i)}) + X^{I}_{0n}(a_{11}^{(i)} - \beta_n^2 a_{14}^{(i)}) + U^{II}_{0n}(a_1^{(i)} - \beta_n^2 a_4^{(i)}) + \tfrac{1}{2}a_n a_2^{(i)}\\
&\quad - \tfrac{1}{2}c_0\beta_n\gamma_n a_4^{(i)} - \tfrac{1}{2}d_0\beta_n\delta_n a_4^{(i)} - \beta_n a_{10}^{(i)} W^{II}_{0n} - \tfrac{1}{2}a_7^{(i)}\beta_n e_n - \tfrac{1}{2}a_{13}^{(i)}\beta_n m_n\\
&\quad + Y^{II}_{0n}(a_{15}^{(i)} - \beta_n^2 a_{18}^{(i)}) + \tfrac{1}{2}q_n a_{16}^{(i)} - \tfrac{1}{2}\beta_n s_0\gamma_n a_{18}^{(i)} - \tfrac{1}{2}t_0\beta_n\delta_n a_{18}^{(i)}\} = 0
\end{aligned} \tag{18d}$$

where the superscript i takes on the values $i = 1, 2, 4$ and 5. Similar operation on eqn (6b) and substitution of eqn (17) will yield:

$$\begin{aligned}
&\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \sin(\alpha_m x_1)\sin(\beta_n x_2)\{-\beta_n a_2^{(3)} U^{I}_{mn} - \alpha_m a_3^{(3)} V^{I}_{mn} + (a_5^{(3)} - \alpha_m^2 a_6^{(3)} - \beta_n^2 a_8^{(3)}) W^{I}_{mn}\\
&\quad - \alpha_m a_9^{(3)} X^{I}_{mn} - \beta_n a_{12}^{(3)} Y^{I}_{mn} - \alpha_m a_1^{(3)} U^{II}_{mn} - \beta_n a_4^{(3)} V^{II}_{mn} + \alpha_m\beta_n a_7^{(3)} W^{II}_{mn}\\
&\quad - \beta_n a_{10}^{(3)} X^{II}_{mn} - \alpha_m a_{11}^{(3)} Y^{II}_{mn}\} = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} q_{mn}\sin(\alpha_m x_1)\sin(\beta_n x_2)
\end{aligned} \tag{19a}$$

$$\begin{aligned}
&\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \cos(\alpha_m x_1)\cos(\beta_n x_2)\{\alpha_m a_1^{(3)} U^{I}_{mn} + \beta_n a_4^{(3)} V^{I}_{mn} + \alpha_m\beta_n a_7^{(3)} W^{I}_{mn} + \beta_n a_{10}^{(3)} X^{I}_{mn}\\
&\quad + \alpha_m a_{11}^{(3)} Y^{I}_{mn} + \beta_n a_2^{(3)} U^{II}_{mn} + a_2^{(3)} c_m\gamma_n + a_2^{(3)} d_m\delta_n + \alpha_m a_3^{(3)} V^{II}_{mn} + a_3^{(3)} e_n\gamma_m + a_3^{(3)} f_n\delta_m\\
&\quad + (a_5^{(3)} - \alpha_m^2 a_6^{(3)} - \beta_n^2 a_8^{(3)}) W^{II}_{mn} + i_n\gamma_m a_6^{(3)} + j_n\delta_m a_6^{(3)} + k_m\gamma_n a_8^{(3)} + l_m\delta_n a_8^{(3)}\\
&\quad + \alpha_m a_9^{(3)} X^{II}_{mn} + a_9^{(3)} m_n\gamma_m + a_9^{(3)} n_n\delta_m + \beta_n a_{12}^{(3)} Y^{II}_{mn} + a_{12}^{(3)} s_m\gamma_n + a_{12}^{(3)} t_m\delta_n\} = 0
\end{aligned} \tag{19b}$$

$$\begin{aligned}
&\sum_{m=1}^{\infty} \cos(\alpha_m x_1)\{\alpha_m a_1^{(3)} U^{I}_{m0} + \alpha_m a_{11}^{(3)} Y^{I}_{m0} + \tfrac{1}{2}c_m a_2^{(3)} + \alpha_m a_3^{(3)} V^{II}_{m0} + \tfrac{1}{2}a_3^{(3)} e_0\gamma_m + \tfrac{1}{2}a_3^{(3)} f_0\delta_m\\
&\quad + (a_5^{(3)} - \alpha_m^2 a_6^{(3)}) W^{II}_{m0} + \tfrac{1}{2}i_0\gamma_m a_6^{(3)} + \tfrac{1}{2}j_0\delta_m a_6^{(3)} + \tfrac{1}{2}a_8^{(3)} k_m + \alpha_m a_9^{(3)} X^{II}_{m0}\\
&\quad + \tfrac{1}{2}a_9^{(3)} m_0\gamma_m + \tfrac{1}{2}a_9^{(3)} n_0\delta_m + \tfrac{1}{2}a_{12}^{(3)} s_m\} = 0
\end{aligned} \tag{19c}$$

$$\sum_{n=1}^{L} \cos(\beta_n x_2)\{\beta_n a_4^{(3)} V'^{\mathrm{I}}_{0n} + \beta_n a_{10}^{(3)} X'^{\mathrm{I}}_{0n} + \beta_n a_2^{(3)} U'^{\mathrm{II}}_{0n} + \tfrac{1}{2} a_2^{(3)} c_0 \gamma_n + \tfrac{1}{2} a_2^{(3)} d_0 \delta_n + \tfrac{1}{2} a_3^{(3)} e_n$$

$$+ (a_5^{(3)} - \beta_n^2 a_8^{(3)}) W'^{\mathrm{II}}_{0n} + \tfrac{1}{2} a_6^{(3)} i_n + \tfrac{1}{2} a_8^{(3)} \gamma_n k_0 + \tfrac{1}{2} a_8^{(3)} \delta_n l_0 + \tfrac{1}{2} a_9^{(3)} m_n + \beta_n a_{12}^{(3)} Y'^{\mathrm{II}}_{0n}$$

$$+ \tfrac{1}{2} a_{12}^{(3)} s_0 \gamma_n + \tfrac{1}{2} a_{12}^{(3)} t_0 \delta_n\} = 0 \tag{19d}$$

$$\tfrac{1}{4} a_2^{(3)} c_0 + \tfrac{1}{4} a_3^{(3)} e_0 + a_5^{(3)} W'^{\mathrm{II}}_{00} + \tfrac{1}{4} a_6^{(3)} i_0 + \tfrac{1}{4} a_8^{(3)} k_0 + \tfrac{1}{4} a_9^{(3)} m_0 + \tfrac{1}{4} a_{12}^{(3)} s_0 = 0. \tag{19e}$$

Equating the coefficients of the trigonometric functions of eqns (18) and (19) will contribute $10mn+5m+5n+1$ linear algebraic equations. The remaining $10m+10n+12$ linear algebraic equations are supplied by the geometric and natural boundary conditions, given by eqns (8). For example, satisfaction of the geometric boundary conditions $u_1(0, x_2) = u_1(a, x_2) = 0$ and equating the coefficients of $\sin(\beta_n x_2)$ will contribute the following $2n$ linear algebraic equations:

$$\sum_{m=1}^{r} \delta_m U'^{\mathrm{II}}_{mn} = 0; \quad U'^{\mathrm{II}}_{0n} + \sum_{m=1}^{r} \gamma_m U'^{\mathrm{II}}_{mn} = 0; \quad \text{for all} \quad n = 1, 2, \ldots. \tag{20}$$

The remaining $10m+8n+12$ equations are contributed by the rest of the boundary conditions of eqns (8). Therefore, depending on the desired degree of accuracy, a finite set of $10mn+15m+15n+13$ linear algebraic equations in as many unknowns needs to be solved. In the interest of computational efficiency, eqns (18), (19) are solved for U'_{mn}, V'_{mn}, W'_{mn}, X'_{mn}, Y'_{mn}, $i = \mathrm{I}, \mathrm{II}$ in terms of the constant coefficients a_n, b_n, c_m, d_m, etc. which are then substituted into the equations arising out of the boundary conditions [e.g. eqns (20)] finally yielding a_n, b_n, c_m, d_m, etc. following an approach suggested by Chaudhuri (1989). This operation will reduce the size of the problem under consideration by an order of magnitude, finally resulting in $10m+10n+12$ linear algebraic equations in as many unknowns, the solutions of which are relatively speaking trivial.

It may be noted that for an antisymmetric angle-ply plate ($R = \infty$),

$$a_3^{(1)} = a_6^{(1)} = a_2^{(2)} = a_8^{(1)} = a_4^{(2)} = a_9^{(1)} = a_1^{(3)} = a_{10}^{(1)} = a_9^{(2)} = a_2^{(3)} = a_3^{(3)} = a_{14}^{(1)} = a_{17}^{(1)}$$

$$= a_{13}^{(2)} = a_{16}^{(2)} = a_4^{(4)} = a_7^{(4)} = a_3^{(5)} = a_6^{(5)} = a_5^{(2)} = a_7^{(2)} = a_{10}^{(2)} = a_4^{(3)} = a_{15}^{(2)}$$

$$= a_5^{(5)} = a_5^{(3)} = a_{10}^{(3)} = a_{11}^{(3)} = a_{10}^{(4)} = a_9^{(5)} = 0. \tag{21}$$

Substitution of eqn (21) into eqns (6) will reduce them to their flat plate counterparts. Furthermore, substitution of eqn (21) into the final solutions [eqns (18)–(20)] can be shown to yield the corresponding flat plate solution presented by Bert and Chen (1978) and Reddy (1984). Substitution of

$$a_3^{(1)} = a_6^{(1)} = a_2^{(2)} = a_8^{(1)} = a_4^{(2)} = a_{10}^{(1)} = a_9^{(2)} = a_2^{(3)} = a_3^{(3)} = a_{13}^{(1)} = a_3^{(4)} = a_{16}^{(1)} = a_{12}^{(2)}$$

$$= a_6^{(4)} = a_2^{(5)} = a_{18}^{(1)} = a_{14}^{(2)} = a_8^{(4)} = a_4^{(5)} = a_7^{(2)} = a_{17}^{(2)} = a_7^{(5)} = a_{10}^{(3)}$$

$$= a_{11}^{(3)} = a_{10}^{(4)} = a_9^{(5)} = 0 \tag{22}$$

will reduce eqns (6) to their counterparts for a homogeneous orthotropic plate—eqns (8), (A1) of Chaudhuri and Kabir (1989), who have presented solutions only for the SS4-type simply-supported and C4-type clamped boundary conditions. However, solution in the form of linear algebraic equations for the SS4-type boundary conditions can easily be modified to obtain their SS2-type counterparts, which can be shown to be identical to those obtained by substitution of eqn (22) into eqns (18)–(20), the details of which will be omitted here in the interest of brevity. This will be further dealt with in Section 5.

4. SOLUTION FOR A CIRCUMFERENTIALLY COMPLETE CYLINDRICAL SHELL

For a circumferentially complete circular cylindrical shell, the x_2 coordinate is replaced by the angular coordinate, θ, where $x_2 = R\theta$. The resulting governing partial differential equations are similar to eqns (6). The assumed solution functions are still given by eqns (9), (10) with the argument $\beta_n x_2$ replaced by $n\theta$, where n is as before, an integer. Since the assumed trigonometric functions of $n\theta$ automatically satisfy the closure condition of a circumferentially complete cylindrical shell, the partial derivatives of all the assumed solution functions with respect to θ can be obtained by termwise differentiation. Only the assumed solution functions with superscript II, and/or their first partial derivatives with respect to x_1 may have edge discontinuities, in which case termwise differentiation will no longer be valid. This will be illustrated for the partial derivatives $u^{II}_{1,1}$ and $u^{II}_{1,11}$ as before:

$$u^{II}_{1,1}(x_1,\theta) = -\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} U^{II}_{mn}\alpha_m \sin(\alpha_m x_1)\sin(n\theta); \quad (0 < x_1 < a) \tag{23a}$$

$$u^{II}_{1,11}(x_1,\theta) = \frac{1}{2}\sum_{m=1}^{\infty} a_n \sin(n\theta) + \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}(-\alpha_m^2 U^{II}_{mn} + a_n\gamma_m + b_n\delta_m)\cos(\alpha_m x_1)\sin(n\theta). \tag{23b}$$

Comparison with the preceding section will reveal that all the unknown coefficients with subscript m, e.g. c_m, d_m, will vanish in the case of a circumferentially complete circular cylindrical shell. Following an identical procedure as in the preceding section, a finite set of $10mn+5m+5n+7$ equations in terms of the identical number of unknowns is generated. Following the procedure given by Chaudhuri (1989), these are reduced to a system of $5m+5n+6$ linear algebraic equations in as many unknowns, a_n, b_n, ..., which can be easily solved.

5. NUMERICAL RESULTS AND DISCUSSIONS

The present study will investigate, as an example, a two-layer antisymmetric angle-ply moderately-thick cylindrical panel of square planform ($a = b$) with fiber orientation of $-\bar{\theta}/\bar{\theta}$. The layers are of equal thickness. The length, a, and the mean radius, R, of the cylindrical panel are 812.8 mm (32 in.) and 2438 mm (96 in.), respectively. The uniformly distributed pressure, q, is 0.6895 GPa (100 ksi). $E_1(=E_L)$ and $E_2(=E_T)$, Young's moduli of a layer in the directions parallel and transverse to the fiber direction, are 172.4 and 6.895 GPa (25,000 and 1000 ksi), respectively. The shear modulus G_{23} $(=G_{TT})$ is assumed to be equal to 3.448 GPa (500 ksi). The shear moduli, G_{12} and G_{13} $(=G_{LT})$, are assumed to be the same in the case of highly crystalline graphite fiber-reinforced composites. The major Poisson's ratio, ν_{12} $(=\nu_{LT})$, is taken equal to 0.25. Sanders' kinematic relations, extended to FSDT, have been used in the computation. This example is selected, because analytical solutions for the corresponding spherical panels of otherwise identical material and geometric parameters have recently become available (Chaudhuri and Abu-Arja, 1988). The following non-dimensionalized quantities are defined:

$$\begin{aligned} u_i^* &= 320E_2h^2u_i/(qa^3); \quad i = 1,2; & u_1^{**} &= 10u_1^* \\ u_3^* &= 320E_2h^3u_3/(qa^4); & \phi_2^* &= 10\phi_2 \\ M_i^* &= 1024M_i/(qa^2); \quad i = 1,2. \end{aligned} \tag{24}$$

Table 1 presents a convergence study of u_3^*, u_1^*, ϕ_1, M_1^* for various aspect ratios, a/h, and fiber orientation angles, $\bar{\theta}$. u_3^* and M_1^* are computed at the center of the panel, while u_1^* and ϕ_1 are computed at the mid-points of the two sides, $(a/2,0)$ and $(0,b/2)$, respectively. It may be noted that not only the side-to-thickness ratio, a/h, but also $\bar{\theta}$ plays a role in the convergence of displacements, rotations and bending moments. Convergence rates of u_3^*,

Table 1. Convergence of u_3^* (at the center), u_1^* (at $x_1 = a/2$, $x_2 = 0$), ϕ_1 (at $x_1 = 0$, $x_2 = b/2$), and M_1^* (at the center) for various aspect ratios, a/h, and fiber orientation angle, θ

	θ		0			22.5		
	a/h	$m = n$	4	6	8	4	6	8
u_3^*	10		2.708	2.738	2.737	3.505	3.547	3.548
	20		1.896	1.923	1.924	2.441	2.484	2.496
u_1^*	10		0	0	0	1.725	1.754	1.762
	20		0	0	0	1.299	1.357	1.367
ϕ_1	10		1.912	1.932	1.932	2.979	3.009	3.019
	20		13.75	13.98	13.99	19.05	19.44	19.57
M_1^*	10		123.3	125.2	124.9	79.63	81.78	81.23
	20		110.0	112.3	112.0	62.25	64.59	64.30

ϕ_2, M_2^* are similar to those of u_1^*, ϕ_1 and M_1^*, respectively, and hence these are not shown here. The convergences for the displacements and rotations of two moderately-thick cylindrical panels shown in Table 1 are reasonably rapid and may be regarded monotonic. The results for the special case of homogeneous orthotropic shells ($\theta = 0^\circ$) compare favorably well with those obtained using an extension of the approach presented by Chaudhuri and Kabir (1989) to the case of SS2-type boundary conditions. These results testify to the validity of the analytical approach and the accuracy of the numerical results. It may be noted that the convergence for the moment, M_1^*, exhibits bounded oscillation, which is acceptable according to the theory of Fourier series, as expounded by Hobson (1926). The amplitude of oscillation, however, decreases with the increase of the number of terms, while the mean value increases, and the amplitude of oscillation serves as an error norm. This phenomenon has also been observed by Chaudhuri and Kabir (1989). Numerical results that are presented below have been obtained by using $m = n = 8$.

Figures 2 and 3 show the variation of the non-dimensionalized displacements, rotations and moments along the centerline, $x_2 = b/2$, of a moderately-thick cylindrical panel ($a/h = 10$) with $\theta = 45^\circ$. Figure 4 exhibits the variation with the fiber orientation angle, θ, of the non-dimensionalized transverse displacement, u_3^*, at the center $(a/2, b/2)$; the surface-parallel displacement, u_1^*, and the rotation, ϕ_2, at the mid-point of a side parallel to the x_1-axis, $(a/2, 0)$; and the surface-parallel displacement, u_2^*, and the rotation, ϕ_1, at the mid-point of a side parallel to the x_2-axis, $(0, b/2)$. The variation of the central moments with fiber orientation angle, θ, is presented in Fig. 5. It may be noted that the transverse displacement, w, and the bending moment, M_1, are symmetric with respect to the centerline

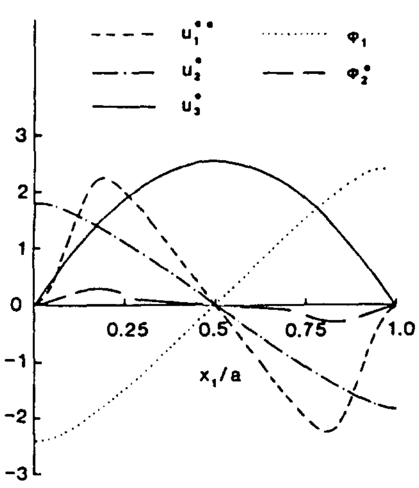

Fig. 2. Variation of displacements and rotations along the center line, $x_2 = b/2$, of a cylindrical panel.

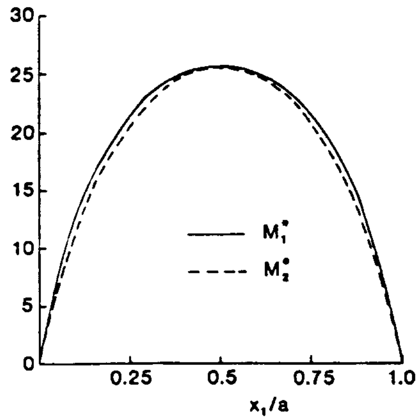

Fig. 3. Variation of bending moments along the center line, $x_2 = b/2$, of a cylindrical panel.

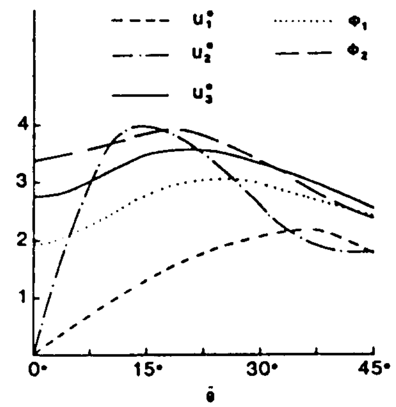

Fig. 4. Variation of displacements and rotations with the fiber orientation angle, $\bar{\theta}$.

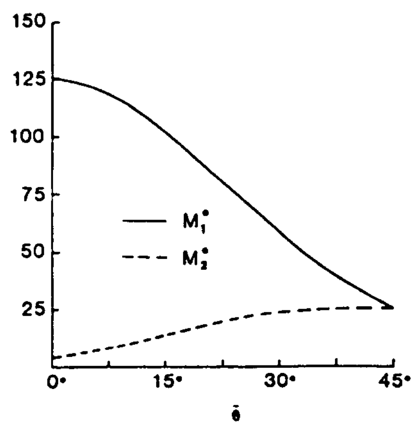

Fig. 5. Variation of bending moments with the fiber orientation angle, $\bar{\theta}$.

$x_1 = a/2$, while the surface-parallel displacements and rotations are antisymmetric with respect to the same. Similarity of these plots to their independently computed spherical [see Figs 2–5, Chaudhuri and Abu-Arja (1988)] counterparts testifies to the accuracy of both the sets of results. Currently unavailable numerical results, on (i) comparison with other shell theories (e.g. Donnell and Love), (ii) comparison of free vibration results with those of Soldatos (1987) and Kabir and Chaudhuri (1991), and (iii) circumferentially complete cylindrical shells, will be published in forthcoming papers.

6. CONCLUSIONS

The analytical (exact in the limit) or strong (or differential) forms of solutions to the problems of moderately-thick antisymmetric angle-ply circular cylindrical panels and circumferentially complete circular cylindrical shells have been obtained utilizing a recently developed novel boundary-discontinuous double Fourier series approach. This method facilitates the well-posedness of the Fourier analysis through selection of the unknown coefficients of the assumed double Fourier series solution functions and introduction of boundary-discontinuous coefficients, so that the number of final algebraic equations become equal to the number of unknowns to furnish a unique complete solution. The two benchmark problems investigated here were hitherto considered by the researchers in the field to be incapable of admitting analytical solutions, which are expected to serve as baselines (in the context of FSDT) for future comparison with those obtained by such popular numerical techniques as the finite element methods. Analytical solutions have been obtained here for three kinematic relations, which are extensions of those due to Sanders, Love and Donnell to the first-order shear deformation theory (FSDT). Numerical results, that have been presented for two-layer antisymmetric angle-ply cylindrical panels of square planform utilizing the extended version of Sanders' kinematic relations, demonstrate good convergence, and also the effects of side-to-thickness ratio and fiber orientation angle on the static response of these panels. Extension of the approach presented herein to (i) include the effect of boundary constraints, (ii) the problem of free vibration and (iii) other types of laminated shells, and currently unavailable numerical results, on (i) comparison with other shell theories (e.g. Donnell and Love), (ii) for free vibration problems and (iii) circumferentially complete cylindrical shells, will be published in forthcoming papers.

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APPENDIX A

The differentiation of full-range double Fourier series

Essential principles of differentiation have been expounded by Hobson in his classic volume (1926) on the subject and reviewed by Winslow (1951), in the context of ordinary Fourier series. Extension of the same to the case of double Fourier series has been treated by Chaudhuri (1989).

In general, the series obtained by differentiating a convergent double Fourier series, representing a function $F(x_1, x_2)$, is not convergent; neither is the series so obtained necessarily the Fourier series corresponding to the particular partial derivative of $F(x_1, x_2)$. Let $F(x_1, x_2)$ be a bounded function and be piecewise continuous (i.e. continuous except for a finite number of ordinary discontinuities); let it also be assumed that the partial derivatives, $F_{,i}(x_1, x_2)$, $i = 1, 2$ have a Lebesgue integral in the domain $(-a, a) \times (-b, b)$ and that if it has lines of infinite discontinuity (e.g. Dirac Delta function), such lines form a reducible set. This is consistent with there being a set of lines of zero measure at which $F_{,i}(x_1, x_2)$ has no definite value.

We consider as an example $u_1^{11}(x_1, x_2)$, which, as defined in eqns (10), is an even function with respect to x_1 and an odd function with respect to x_2,

$$u_1^{11}(x_1, -x_2) = -u_1^{11}(x_1, x_2) = -u_1^{11}(-x_1, x_2). \tag{A1}$$

The full-range double Fourier series expansion for the function and its two first partial derivatives are as follows:

$$u_1^{11}(x_1, x_2) = \sum_{m=0}^{\infty} \sum_{n=1}^{\infty} U_{mn}^{11} \cos(\alpha_m x_1) \sin(\beta_n x_2) \tag{A2a}$$

$$u_{1,1}^{11}(x_1, x_2) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} B_{mn}^{11} \sin(\alpha_m x_1) \sin(\beta_n x_2) \tag{A2b}$$

$$u_{1,2}^{11}(x_1, x_2) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} C_{mn}^{11} \cos(\alpha_m x_1) \cos(\beta_n x_2) \tag{A2c}$$

wherein

$$U_{mn}^{11} = \frac{1}{ab} \int_{-a}^{a} \int_{-b}^{b} u_1^{11}(x_1, x_2) \cos(\alpha_m x_1) \sin(\beta_n x_2) \, dx_1 \, dx_2 \quad \text{for} \quad m, n = 1, 2, \ldots, \infty \tag{A3a}$$

$$U_{0n}^{11} = \frac{1}{2ab} \int_{-a}^{a} \int_{-b}^{b} u_1^{11}(x_1, x_2) \sin(\beta_n x_2) \, dx_1 \, dx_2 \quad \text{for} \quad n = 1, 2, \ldots, \infty \tag{A3b}$$

$$B_{mn}^{\text{II}} = \frac{1}{ab}\int_{-a}^{a}\int_{-b}^{b} u_{1,1}^{\text{II}}(x_1, x_2)\sin(\alpha_m x_1)\sin(\beta_n x_2)\,\mathrm{d}x_1\,\mathrm{d}x_2 \quad \text{for} \quad m, n = 1, 2, \ldots, \infty \tag{A3c}$$

$$C_{mn}^{\text{II}} = \frac{1}{ab}\int_{-a}^{a}\int_{-b}^{b} u_{1,2}^{\text{II}}(x_1, x_2)\cos(\alpha_m x_1)\cos(\beta_n x_2)\,\mathrm{d}x_1\,\mathrm{d}x_2 \quad \text{for} \quad m, n = 1, 2, \ldots, \infty \tag{A3d}$$

$$C_{0n}^{\text{II}} = \frac{1}{2ab}\int_{-a}^{a}\int_{-b}^{b} u_{1,2}^{\text{II}}(x_1, x_2)\cos(\beta_n x_2)\,\mathrm{d}x_1\,\mathrm{d}x_2 \quad \text{for} \quad n = 1, 2, \ldots, \infty \tag{A3e}$$

$$C_{m0}^{\text{II}} = \frac{1}{2ab}\int_{-a}^{a}\int_{-b}^{b} u_{1,2}^{\text{II}}(x_1, x_2)\cos(\alpha_m x_1)\,\mathrm{d}x_1\,\mathrm{d}x_2 \quad \text{for} \quad m = 1, 2, \ldots, \infty \tag{A3f}$$

$$C_{00}^{\text{II}} = \frac{1}{4ab}\int_{-a}^{a}\int_{-b}^{b} u_{1,2}^{\text{II}}(x_1, x_2)\,\mathrm{d}x_1\,\mathrm{d}x_2. \tag{A3g}$$

Integration by parts of r.h.s. of eqn (A3a) will yield the following:

$$B_{mn}^{\text{II}} = -\alpha_m U_{mn}^{(2)} + \frac{1}{ab}\int_{-b}^{b}\left[\sum_{d=1}^{d^{(N1)}} \{u_1^{\text{II}}(x_{1d}-0, x_2) - u_1^{\text{II}}(x_{1d}+0, x_2)\}\sin(\alpha_m x_{1d})\right] \times \sin(\beta_n x_2)\,\mathrm{d}x_2 \quad \text{for} \quad m, n = 1, 2, \ldots, \infty \tag{A4a}$$

$$C_{mn}^{\text{II}} = \beta_n U_{mn}^{\text{II}} + \frac{1}{ab}\int_{-a}^{a} \{u_1^{\text{II}}(x_1, b-0) - u_1^{\text{II}}(x_1, -b+0)\}(-1)^n\cos(\alpha_m x_1)\,\mathrm{d}x_1 + \frac{1}{ab}\int_{-a}^{a}\sum_{d=1}^{d^{(N2)}} \{u_1^{\text{II}}(x_1, x_{2d}-0) - u_1^{\text{II}}(x_1, x_{2d}+0)\}\sin(\beta_n x_{2d})\cos(\alpha_m x_1)\,\mathrm{d}x_1 \quad \text{for} \quad m, n = 1, 2, \ldots, \infty \tag{A4b}$$

$$C_{m0}^{\text{II}} = \frac{1}{2ab}\int_{-a}^{a} \{u_1^{\text{II}}(x_1, b-0) - u_1^{\text{II}}(x_1, -b+0)\}\cos(\alpha_m x_1)\mathrm{d}x_1 \quad \textit{for} \quad m = 1, 2, \ldots, \infty \tag{A4c}$$

$$C_{0n}^{\text{II}} = \beta_n U_{0n}^{\text{II}} + \frac{1}{2ab}\int_{-a}^{a} \{u_1^{\text{II}}(x_1, b-0) - u_1^{\text{II}}(x_1, -b+0)\}(-1)^n\,\mathrm{d}x_1 \quad \text{for} \quad n = 1, \ldots, \infty \tag{A4d}$$

$$C_{00}^{\text{II}} = \frac{1}{4ab}\int_{-a}^{a} \{u_1^{\text{II}}(x_1, b-0) - u_1^{\text{II}}(x_1, -b+0)\}\,\mathrm{d}x_1 \tag{A4e}$$

$d^{(Ni)}$, $i = 1, 2$, in eqns (A4a), (A4b) represents the number of discontinuities in the direction, x_i.

It is worthwhile to draw attention to a theorem [see Hobson (1926) and Chaudhuri (1989)] which implies that if a function $F(x_1, x_2)$ is an odd function of x_i, $i = 1, 2$, represented by a full-range double Fourier series in the domain $(-a, a) \times (-b, b)$, termwise partial differentiation of the Fourier series with respect to x_i does not represent the first partial derivative $F_{,i}(x_1, x_2)$ unless (i) $F_{,i}(x_1, x_2)$ is continuous in the interior of the above domain, and (ii)

$$F(a-0, x_2) = F(-a+0, x_2) = 0 \quad \text{for} \quad i = 1 \text{ and/or} \quad F(x_1, b-0) = F(x_1, -b+0) \quad \text{for} \quad i = 2. \tag{A5}$$

In the case of an even function, condition (i) will suffice. This is clearly true for $u_1^{\text{II}}(x_1, x_2)$ and its first partial derivatives, as is evident from eqns (A4).

Differentiation of half-range double Fourier series

The present solution, e.g. $u_1^{\text{II}}(x_1, x_2)$ [see eqns (10)], is represented by double Fourier series in the domain $(0, a) \times (0, b)$, the lengths of the intervals in the directions, x_1 and x_2, being one-half of the full ranges of intervals of periodicity, $2a$ and $2b$, respectively. $u_1^{\text{II}}(x_1, x_2)$, in addition to being an even function of x_1 and an odd function of x_2 as stated earlier, is also continuous in the interior of the domain $(0, a) \times (0, b)$, and does not vanish at the edges, $x_2 = 0, b$. The half-range double Fourier series representation for $u_1^{\text{II}}(x_1, x_2)$ and its first two partial derivatives are given by eqns (A1), wherein

$$U_{mn}^{\text{II}} = \frac{4}{ab}\int_0^a\int_0^b u_1^{\text{II}}(x_1, x_2)\cos(\alpha_m x_1)\sin(\beta_n x_2)\,\mathrm{d}x_1\,\mathrm{d}x_2 \quad \text{for} \quad m, n = 1, 2, \ldots, \infty \tag{A6a}$$

$$B_{mn}^{\text{II}} = \frac{4}{ab}\int_0^a\int_0^b u_{1,1}^{\text{II}}(x_1, x_2)\sin(\alpha_m x_1)\sin(\beta_n x_2)\,\mathrm{d}x_1\,\mathrm{d}x_2 \quad \text{for} \quad m, n = 1, 2, \ldots, \infty \tag{A6b}$$

$$C_{mn}^{\text{II}} = \frac{4}{ab}\int_0^a\int_0^b u_{1,2}^{\text{II}}(x_1, x_2)\cos(\alpha_m x_1)\cos(\beta_n x_2)\,\mathrm{d}x_1\,\mathrm{d}x_2 \quad \text{for} \quad m, n = 1, 2, \ldots, \infty \tag{A6c}$$

$$C_{m0}^{\text{II}} = \frac{2}{ab}\int_0^a\int_0^b u_{1,2}^{\text{II}}(x_1, x_2)\cos(\alpha_m x_1)\,\mathrm{d}x_1\,\mathrm{d}x_2 \quad \text{for} \quad m = 1, \ldots, \infty \tag{A6d}$$

$$C_{0n}^{\text{II}} = \frac{2}{ab}\int_0^a\int_0^b u_{1,2}^{\text{II}}(x_1, x_2)\cos(\beta_n x_2)\,\mathrm{d}x_1\,\mathrm{d}x_2 \quad \text{for} \quad n = 1, 2, \ldots, \infty \tag{A6e}$$

$$C_{00}^{II} = \frac{1}{ab}\int_0^a \int_0^b u_{1,2}^{II}(x_1, x_2)\,dx_1\,dx_2. \tag{A6f}$$

Substitution of

$$d^{(x1)} = 1; \quad x_{1d} = 0 \quad \text{and} \quad u_1^{II}(0-0, x_2) = u_1^{II}(0+0, x_2) \tag{A7}$$

into eqn (A4a) will yield

$$B_{mn} = -\alpha_m U_{mn}^{II}, \tag{A8}$$

which implies that the first partial derivative, $u_{1,1}^{II}(x_1, x_2)$, can be obtained by termwise differentiation of the half-range Fourier series expansion of $u_1^{II}(x_1, x_2)$, represented by eqns (A1a), (A6a). However, the other first partial derivative, $u_{1,2}^{II}(x_1, x_2)$, cannot be represented by the termwise differentiation of the series, given by eqns (A1a) and (A6a), because $u_{1,2}^{II}(x_1, x_2)$ has ordinary discontinuity at the line $x_2 = 0$ and also because

$$u_1^{II}(x_1, b-0) \neq u_1^{II}(x_1, -b+0) \neq 0. \tag{A9}$$

The Fourier coefficients for $u_{1,2}^{II}(x_1, x_2)$ must then be obtained by substituting in eqns (A4)

$$d^{(x2)} = 1; \quad x_{2d} = 0 \quad \text{and} \quad u_1^{II}(x_1, -b+0) = -u_1^{II}(x_1, b-0); \quad u_1^{II}(x_1, 0-0) = -u_1^{II}(x_1, 0+0) \tag{A10}$$

which will finally yield

$$C_{mn}^{II} = \beta_n U_{mn}^{II} + \frac{4}{ab}\int_0^a \{u_1^{II}(x_1, b-0)(-1)^n - u_1^{II}(x_1, 0+0)\}\cos(\alpha_m x_1)\,dx_1 \quad \text{for} \quad m, n = 1, 2, \ldots, \infty \tag{A11a}$$

$$C_{m0}^{II} = \beta_n U_{mn}^{II} + \frac{2}{ab}\int_0^a \{u_1^{II}(x_1, b-0) - u_1^{II}(x_1, 0+0)\}\cos(\alpha_m x_1)\,dx_1 \quad \text{for} \quad m = 1, 2, \ldots, \infty \tag{A11b}$$

$$C_{0n}^{II} = \frac{2}{ab}\int_0^a \{u_1^{II}(x_1, b-0) - u_1^{II}(x_1, 0+0)\}\cos(\alpha_m x_1)\,dx_1 \quad \text{for} \quad n = 1, 2, \ldots, \infty \tag{A11c}$$

$$C_{00}^{II} = \frac{1}{ab}\int_0^a \{u_1^{II}(x_1, b-0) - u_1^{II}(x_1, 0+0)\}\,dx_1. \tag{A11d}$$

APPENDIX B

Definition of certain constants

The non-zero constants referred to in eqn (6) are written as follows:

$$\begin{gathered}
a_2^{(1)} = A_{11}; \quad a_3^{(1)} = -2\bar{c}_2 B_{16}/R; \quad a_4^{(1)} = a_6^{(2)} = A_{66} + \bar{c}_2^2 D_{66}/R^2; \quad a_6^{(1)} = a_2^{(2)} = \bar{c}_2 B_{16}/R; \\
a_7^{(1)} = a_3^{(2)} = A_{12} + A_{66} - \bar{c}_2^2 D_{66}/R^2; \quad a_8^{(1)} = a_4^{(2)} = -\bar{c}_2 B_{26}/R; \quad a_9^{(1)} = a_1^{(3)} = A_{12}/R; \\
a_{10}^{(1)} = -a_9^{(2)} = a_2^{(3)} = -a_3^{(3)} = -\bar{c}_2 B_{26}/R^2; \quad a_{11}^{(1)} = a_3^{(4)} = 2B_{16}; \\
a_{14}^{(1)} = a_{17}^{(1)} = -a_{13}^{(2)} = -a_{16}^{(2)} = a_4^{(4)} = -a_7^{(4)} = a_3^{(5)} = -a_6^{(5)} = -\bar{c}_2 D_{66}/R; \quad a_{16}^{(1)} = a_{12}^{(2)} = a_6^{(4)} = a_2^{(5)} = B_{16}; \\
a_{18}^{(1)} = a_{14}^{(2)} = a_8^{(4)} = a_4^{(5)} = B_{26}; \quad a_5^{(2)} = -\bar{c}_1^2 A_{44}/R^2; \quad a_7^{(2)} = 2\bar{c}_2 B_{26}/R; \quad a_8^{(2)} = A_{22}; \\
a_{10}^{(2)} = a_4^{(3)} = (A_{22} + \bar{c}_1 A_{44})/R; \quad a_{15}^{(2)} = a_5^{(3)} = \bar{c}_2 A_{44}/R; \quad a_{17}^{(2)} = a_7^{(3)} = 2B_{26}; \quad a_5^{(3)} = A_{22}/R^2; \\
a_6^{(3)} = a_9^{(3)} = a_9^{(4)} = a_{11}^{(4)} = -A_{55}; \quad a_8^{(3)} = a_{12}^{(3)} = a_{10}^{(5)} = a_{15}^{(5)} = -A_{44}; \quad a_{10}^{(3)} = a_{11}^{(3)} = a_{10}^{(4)} = a_9^{(5)} = B_{26}/R; \\
a_{12}^{(4)} = D_{11}; \quad a_{14}^{(4)} = a_{16}^{(5)} = D_{66}; \quad a_{17}^{(4)} = a_{13}^{(5)} = D_{12} + D_{66}; \quad a_{18}^{(5)} = D_{22}.
\end{gathered} \tag{B1}$$